

ON THE SYZYGIES AND ALEXANDER POLYNOMIALS OF NODAL HYPERSURFACES

ALEXANDRU DIMCA¹ AND GABRIEL STICLARU

ABSTRACT. We give sharp lower bounds for the degree of the syzygies involving the partial derivatives of a homogeneous polynomial defining a nodal hypersurface. The result gives information on the position of the singularities of a nodal hypersurface expressed in terms of defects or superabundances.

The case of Chebyshev hypersurfaces is considered as a test for this result and leads to a potentially infinite family of nodal hypersurfaces having nontrivial Alexander polynomials.

1. INTRODUCTION

Let $S = \mathbb{C}[x_0, \dots, x_n]$ be the graded ring of polynomials in x_0, \dots, x_n with complex coefficients and denote by S_r the vector space of homogeneous polynomials in S of degree r . For any polynomial $f \in S_r$, we define the *Jacobian ideal* $J_f \subset S$ as the ideal spanned by the partial derivatives f_0, \dots, f_n of f with respect to x_0, \dots, x_n and the corresponding graded *Milnor* (or *Jacobian*) *algebra* by

$$(1.1) \quad M(f) = S/J_f.$$

The study of such Milnor algebras is related to the singularities of the corresponding projective hypersurface $D : f = 0$, see [3], as well as to the mixed Hodge theory of the hypersurface D and of its complement $U = \mathbb{P}^n \setminus D$, see the foundational article by Griffiths [12] and also [4], [6], [7], [9].

The Milnor algebra $M(f)$ can be seen (up to a twist in grading) as the top cohomology of the Koszul complex $K^*(f)$ of the partial derivatives f_0, \dots, f_n in S , see [3] or [5], Chapter 6. As such, it is related to certain natural E_1 -spectral sequences associated to the pole order filtration and converging to the cohomology of the complement U introduced in [4], discussed in detail in [5], Chapter 6 and reconsidered recently in [9].

In the second section we study one of these spectral sequences for nodal hypersurfaces, using a key result by M. Saito telling when the Hodge filtration coincide to the pole order filtration on the cohomology groups $H^*(U)$. This study gives sharp lower bounds for the degree of the syzygies involving the partial derivatives of a homogeneous polynomial defining a nodal hypersurface, extending the result proved

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in the curve case in Theorem 4.1 in [9] to arbitrary dimension. In the curve case, see also [14] and [11].

In the third section we consider the special case of Chebyshev hypersurfaces, which are classical examples of nodal hypersurfaces with many singularities. They were introduced by S. V. Chmutov to construct complex projective hypersurfaces with a large number of nodes, i.e. A_1 -singularities, see [1], volume 2, p. 419 and [2].

In the final section, we show that on one hand the lower bounds obtained in the general case are best possible for curves and 3-dimensional Chebyshev hypersurfaces of degree ≤ 20 (and probably for all odd dimensional Chebyshev hypersurfaces, see Conjecture 3.3), and on the other hand we give some topological applications, by computing the Alexander polynomials of Chebyshev hypersurfaces of dimension 2 and 3 and degree $d \leq 20$.

The Alexander polynomials of singular hypersurfaces were introduced by A. Libgober [15], [16] and are very subtle invariants of the topology of the complement U . However the number of classes of hypersurfaces where these Alexander polynomials are not trivial is rather limited, and this explains the interest of our new examples.

To end this Introduction, we recall the following notions, introduced in [9].

Definition 1.1. For a hypersurface $D : f = 0$ with isolated singularities we introduce three integers, as follows:

(i) the *coincidence threshold* $ct(D)$ defined as

$$ct(D) = \max\{q : \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \leq q\},$$

with f_s a homogeneous polynomial in S of degree $d = \deg f$ such that $D_s : f_s = 0$ is a smooth hypersurface in \mathbb{P}^n .

(ii) the *stability threshold* $st(D)$ defined as

$$st(D) = \min\{q : \dim M(f)_k = \tau(D) \text{ for all } k \geq q\}$$

where $\tau(D)$ is the total Tjurina number of D , i.e. the sum of all the Tjurina numbers of the singularities of D .

(iii) the *minimal degree of a nontrivial syzygy* $mdr(D)$ defined as

$$mdr(D) = \min\{q : H^n(K^*(f))_{q+n} \neq 0\}$$

where $K^*(f)$ is the Koszul complex of f_0, \dots, f_n with the natural grading defined in [9].

Moreover it is easy to see that one has

$$(1.2) \quad ct(D) = mdr(D) + d - 2.$$

Recall also that, for a finite set of points $\mathcal{N} \subset \mathbb{P}^n$, we denote by

$$\text{def } S_m(\mathcal{N}) = |\mathcal{N}| - \text{codim}\{h \in S_m \mid h(a) = 0 \text{ for any } a \in \mathcal{N}\},$$

the *defect (or superabundance) of the linear system of polynomials in S_m vanishing at the points in \mathcal{N}* , see [5], p. 207. This positive integer is called the *failure of \mathcal{N} to impose independent conditions on homogeneous polynomials of degree m* in [10].

When D is a degree d nodal hypersurface in \mathbb{P}^n , with \mathcal{N} as singular set, it follows from Theorem 1.5 in [9] that one has

$$(1.3) \quad \text{def } S_k(\mathcal{N}) \neq 0 \text{ for } k < T - ct(D) \text{ and } \text{def } S_k(\mathcal{N}) = 0 \text{ for } k \geq T - ct(D)$$

and also

$$(1.4) \quad \text{def } S_k(\mathcal{N}) = |\mathcal{N}| - \dim S_k \text{ for } k \leq T - st(D)$$

where $T = (n+1)(d-2)$.

Note that computing the Hilbert-Poincaré series of the Milnor algebra $M(f)$ using an appropriate software is much easier than computing the defects $\text{def } S_k(\mathcal{N})$, because the Jacobian ideal comes with a given set of $(n+1)$ generators f_0, \dots, f_n , while the ideal I of polynomials vanishing on \mathcal{N} has not such a given generating set. However, it is the defects $\text{def } S_k(\mathcal{N})$, who describe the position of the singularities of D in \mathbb{P}^n and which occur in many geometric problems, see for instance Theorem 4.1 below.

Numerical experiments with the CoCoA package [18] and the Singular package [19] have played a key role in the completion of this work.

2. THE SPECTRAL SEQUENCE AND THE SYZYGIES OF NODAL HYPERSURFACES

Let $D : f = 0$ be a nodal hypersurface in \mathbb{P}^n of degree d .

We consider first the case when $n = 2n_1 + 1 \geq 3$ is odd. Then D is a \mathbb{Q} -homology manifold satisfying $b_j(D) = b_j(D_s)$ for $j \neq n-1$, and the middle Betti number $b_{n-1}(D)$ is computable, e.g. using the formula $b_{n-1}(D) = b_{n-1}(D_s) - n(D)$, where $n(D) = \tau(D)$ is the cardinal of the set \mathcal{N} of nodes of D . It follows that the complement U has at most two non-zero cohomology groups. The first of them, $H^0(U)$ is 1-dimensional and of Hodge type $(0,0)$, so nothing interesting here. The second one, $H^n(U)$, is dual to $H_c^n(U)(-n)$ and $H_c^n(U)$ is isomorphic to $\text{coker}(H^{n-1}(\mathbb{P}^n) \rightarrow H^{n-1}(D))$, the morphism being induced by the inclusion $i : D \rightarrow \mathbb{P}^n$.

It follows that the mixed Hodge structure (for short MHS) on $H^n(U)$ is pure of weight $n+1$ with

$$h^{p,q}(H^n(U)) = h^{p-1,q-1}(D_s),$$

for $p+q = n+1 = 2n_1+2$, $p \neq q$, and

$$h^{n_1+1,n_1+1}(H^n(U)) = h^{n_1,n_1}(D) - 1 = h^{n_1,n_1}(D_s) - n(S) - 1.$$

In particular, we have $P^1 H^n(U) = F^1 H^n(U) = H^n(U)$.

But we have much more than this. Let $\alpha_D = \frac{n}{2}$. Then Corollary (0.12) in M. Saito [17], or even better, the formula (1.1.3) in [7], imply that

$$(2.1) \quad F^s H^n(U) = P^s H^n(U) \text{ for } s \geq n - \alpha_D + 1$$

i.e. for $s \geq n_1 + 2$.

Now we look at the nonzero terms in the E_1 -term of the spectral sequence $E_r^{p,q}(f)$ introduced in [9], Proposition 2.2. Since D has only isolated singularities, these terms are sitting on two lines, given by $L : p+q = n$ and $L' : p+q = n-1$.

We look first at the terms on the line L . The term $E_1^{n-q,q}(f) = H^{n+1}(K^*(f))_{(q+1)d}$ is isomorphic as a \mathbb{C} -vector space to $M(f)_{(q+1)d-n-1}$, see Proposition (2.2) in [9].

The corresponding limit term $E_\infty^{n-q,q}(f) = Gr_P^{n-q}H^n(U)$ coincides to $Gr_F^{n-q}H^n(U)$ in view of (2.1), for $q = 0, \dots, n_1 - 1 = \lfloor \frac{n}{2} \rfloor - 1$, where $[y]$ denotes the integral part of the real number y .

On the other hand, Theorem 2.2 in [7] yields the following isomorphism of \mathbb{C} -vector spaces

$$(2.2) \quad Gr_F^{n-q}H^n(U) = M(f)_{(q+1)d-n-1} \text{ for } q < \left\lfloor \frac{n}{2} \right\rfloor.$$

It follows that in this range we have in fact

$$(2.3) \quad E_1^{n-q,q}(f) = E_\infty^{n-q,q}(f).$$

Therefore all the differentials in the E_1 -spectral sequence $E_r(f)$ arriving at terms $E_r^{n-q,q}(f)$ having $q < \lfloor \frac{n}{2} \rfloor$ are trivial.

We look now at the terms on the line L' . The term $E_1^{n-1-q,q}(f)$ is given by $H^n(K^*(f))_{(q+1)d}$ and the limit term $E_\infty^{n-1-q,q}(f)$ has to be zero, as $H^{n-1}(U) = 0$ for n odd. It follows that

$$(2.4) \quad H^n(K^*(f))_{qd} = 0 \text{ for } q \leq \left\lfloor \frac{n}{2} \right\rfloor = n_1.$$

We discuss now the case $n = 2n_1 \geq 2$ even. Then D is no longer a \mathbb{Q} -homology manifold, but one knows that $b_j(D) = b_j(D_s)$ for $j \notin \{n-1, n\}$, and the n -th Betti number $b_n(D)$ is computable in terms of defects of linear series, namely

$$(2.5) \quad b_n(D) = \text{def } S_{n_1d-2n_1-1}(\mathcal{N}) + 1,$$

see Theorem (6.4.5) on page 208 in [5]. Moreover, the proof implies that the group $H^n(D)$ is a pure Hodge structure of type (n_1, n_1) , and the same holds for $H^{n-1}(U)$.

We consider the hypersurface \tilde{D} in \mathbb{P}^{n+1} given by the equation $\tilde{f}(x_0, \dots, x_{n+1}) = f(x_0, \dots, x_n) + x_{n+1}^d = 0$ and the complement $\tilde{U} = \mathbb{P}^{n+1} \setminus \tilde{D}$. Now \tilde{D} is a \mathbb{Q} -homology manifold, and as above we define $\alpha_{\tilde{D}} = \frac{n}{2} + \frac{1}{d}$. Then the formula (1.1.3) in [7], imply that

$$(2.6) \quad F^s H^{n+1}(\tilde{U}) = P^s H^{n+1}(\tilde{U}) \text{ for } s \geq n_1 + 2.$$

Now we look at the nonzero terms in the E_1 -term of the spectral sequence $E_r^{p,q}(\tilde{f})$. They are sitting on two lines, given by $L : p + q = n + 1$ and $L' : p + q = n$.

We look first at the terms on the line L . The term $E_1^{n+1-q,q}(\tilde{f}) = H^{n+2}(K^*(\tilde{f}))_{(q+1)d}$ is isomorphic as a \mathbb{C} -vector space to $M(\tilde{f})_{(q+1)d-n-2}$, see Proposition (2.2) in [9].

The limit term $E_\infty^{n+1-q,q}(\tilde{f}) = Gr_P^{n+1-q}H^{n+1}(\tilde{U})$ coincides to $Gr_F^{n+1-q}H^{n+1}(\tilde{U})$ in view of (2.6), for $q = 0, \dots, n_1 - 1 = \lfloor \frac{n}{2} \rfloor - 1$.

On the other hand, Theorem 2.2 in [7] yields the following isomorphism of \mathbb{C} -vector spaces

$$(2.7) \quad Gr_F^{n+1-q}H^{n+1}(\tilde{U}) = M(\tilde{f})_{(q+1)d-n-2} \text{ for } q < [\alpha_{\tilde{D}}] = n_1.$$

It follows that in this range we have in fact

$$(2.8) \quad E_1^{n+1-q,q}(\tilde{f}) = E_\infty^{n+1-q,q}(\tilde{f}).$$

All the differentials in the E_1 -spectral sequence $E_r(\tilde{f})$ arriving at terms $E_r^{n+1-q,q}(\tilde{f})$ having $q < \left\lfloor \frac{n}{2} \right\rfloor$ should therefore be trivial.

Let us look now at the terms on the line L' . The term $E_1^{n-q,q}(\tilde{f})$ is given by $H^{n+1}(K^*(\tilde{f}))_{(q+1)d}$ and the limit term $E_\infty^{n-q,q}(\tilde{f})$ has to be zero, as $H^n(\tilde{U}) = 0$ for n even. It follows that

$$(2.9) \quad H^{n+1}(K^*(\tilde{f}))_{qd} = 0 \text{ for } q \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Now a non-zero class $\omega \in H^n(K^*(f))_m$ gives rise to a non-zero class $\tilde{\omega} = \omega \wedge dx_{n+1} \in H^{n+1}(K^*(\tilde{f}))_{m+1}$. It follows that

$$(2.10) \quad H^n(K^*(f))_{qd-1} = 0 \text{ for } q \leq \left\lfloor \frac{n}{2} \right\rfloor = n_1.$$

Now recall that if the coordinates x_0, \dots, x_n are chosen such that the hyperplane at infinity $H_0 : x_0 = 0$ is transversal to D , then the multiplication by x_0 induces an injection $H^n(K^*(f))_{m-1} \rightarrow H^n(K^*(f))_m$ (the dual statement for the homology is part of Corollary 11 in [3]).

This yields our first main result.

Theorem 2.1. *Let $D : f = 0$ be a nodal surface in \mathbb{P}^n of degree d .*

- (i) *If $n = 2n_1 + 1$ is odd, then $H^n(K^*(f))_m = 0$ for any $m \leq n_1 d$.*
- (ii) *If $n = 2n_1$ is even, then $H^n(K^*(f))_m = 0$ for any $m \leq n_1 d - 1$.*

If a single formula is preferred, then $H^n(K^*(f))_m = 0$ for any

$$m \leq \left\lfloor \frac{n}{2} \right\rfloor d - \frac{1 + (-1)^n}{2}.$$

Using the formula (1.2) we get the following.

Corollary 2.2. *Let $D : f = 0$ be a nodal surface in \mathbb{P}^n of degree d and let \mathcal{N} denote its set of nodes.*

- (i) *If $n = 2n_1 + 1$ is odd, then $ct(D) \geq (n_1 + 1)d - n - 1 = \frac{T}{2}$ and $\text{def } S_k(\mathcal{N}) = 0$ for $k \geq (n_1 + 1)d - n - 1 = \frac{T}{2}$.*
- (ii) *If $n = 2n_1$ is even, then $ct(D) \geq (n_1 + 1)d - n - 2 = \frac{T}{2} + (\frac{d-2}{2})$ and $\text{def } S_k(\mathcal{N}) = 0$ for $k \geq n_1 d - n = \frac{T}{2} - (\frac{d-2}{2})$.*

Example 2.3. (i) In the plane curve case, we have $n = 2$, hence (ii) in the above Theorem becomes $H^2(K^*(f))_m = 0$ for any $m \leq d - 1$, which is exactly the first claim in Theorem 4.1 in [9]. Moreover this bound is strict, since $H^2(K^*(f))_d \neq 0$ for reducible curves as shown in loc.cit..

(ii) Corollary 2.2 (i) is a generalization of Theorem 5.1 (iv) in [9]. However, the formula in Remark 4.6 suggests that one should have the sharper bound

$$ct(D) \geq \frac{T}{2} + \left(\frac{d-2}{2}\right)$$

for n odd as well.

3. CHEBYSHEV HYPERSURFACES: STABILITY AND COINCIDENCE THRESHHOLDS

The d -th Chebyshev polynomial $T_d(x) = \cos(d \arccos(x))$ has $d - 1$ critical points, namely $\lambda_k = \cos(k\pi/d)$ for $k = 1, \dots, d - 1$. One has $T_d(\lambda_k) = (-1)^k$.

It follows that, for $d = 2m + 1$ odd, the critical values ± 1 are both attained m times. When $d = 2m$, the maximal critical value 1 is attained $m - 1$ times, while the minimal critical value -1 is attained m times.

Consider the hypersurface $C(n, d, k)$ in \mathbb{P}^n for $n \geq 2$ defined by the homogeneous equation $f(n, d, k) = 0$, where the polynomial $f(n, d, k)$ is the homogenization (using x_0) of the polynomial

$$(3.1) \quad g(n, k, d) = T_d(x_1) + \dots + T_d(x_n) + k.$$

It is easy to see that the hypersurface $C(n, d, k)$ is smooth, unless k is an integer satisfying $|k| \leq n$ and $n + k$ is even. If these two conditions are fulfilled, then the hypersurface $C(n, d, k)$ is nodal, and the number of nodes is

$$\tau(C(n, d, k)) = \binom{n}{a} d_1^n$$

if $d = 2d_1 + 1$, with $2a = n + k$, and

$$\tau(C(n, d, k)) = \binom{n}{a} d_1^n \left(1 - \frac{1}{d_1}\right)^a$$

if $d = 2d_1$, with $2a = n + k$.

It follows that for d odd the maximal number of nodes is obtained for $a = \lfloor \frac{n}{2} \rfloor$. When $n = 2n_1$ is even, this implies that $k = 0$, so in this case the *Chebyshev hypersurface* $\mathcal{C}(n, d)$ corresponds to $k = 0$. For n odd, both values $k = \pm 1$ give the same number of nodes. We pick the value $k = 1$, for the reason explained below.

For d even, it is not clear for which k the maximum of $\tau(C(n, d, k))$ is attained. However, one may show that for $d \geq n + 2$, the maximum is again attained for $a = \lfloor \frac{n}{2} \rfloor$. We will call in this case the *Chebyshev hypersurface* $\mathcal{C}(n, d)$ the hypersurface corresponding to $a = \lfloor \frac{n}{2} \rfloor$, $k = 0$ for n even, and $k = 1$ for n odd. In the latter case, i.e. d even and n odd, the choice $k = -1$ gives a lower number of nodes and it is less interesting, see Remark 4.6 (ii).

In conclusion, the (affine part of) Chebyshev hypersurface $\mathcal{C}(n, d)$ is defined by the affine equation

$$g(n, d) = T_d(x_1) + \dots + T_d(x_n) = 0$$

when n is even, and by

$$g(n, d) = T_d(x_1) + \dots + T_d(x_n) + 1 = 0$$

when n is odd.

Let $\mathcal{N}(n, d)$ be the set of nodes of the Chebyshev hypersurfaces $\mathcal{C}(n, d)$. We may consider this set as a subset of the affine space $\mathbb{C}^n \subset \mathbb{P}^n$ (given by $x_0 = 1$). For $n = 2n_1$ even, the set $\mathcal{N}(n, d)$ is the set of points $a = (a_1, \dots, a_n)$ such that n_1 among

the a_j 's are local minimum points for T_d , and the remaining n_1 coordinates are local maximum points for T_d . When $n = 2n_1 + 1$, we have a similar description, the number of coordinates equal to local minima being $n_1 + 1$, and those of local maxima being n_1 .

Consider the evaluation map

$$ev(n, d)_{\leq r} : \mathbb{C}[x_1, \dots, x_n]_{\leq r} \rightarrow \mathcal{F}(\mathcal{N}(n, d))$$

where $\mathbb{C}[x_1, \dots, x_n]_{\leq r}$ denotes the vector space of polynomials of degree at most r , $\mathcal{F}(\mathcal{N}(n, d))$ denotes the vector space of \mathbb{C} -valued functions on the set $\mathcal{N}(n, d)$, and a polynomial h is mapped to the function sending $a \in \mathcal{N}(n, d)$ to $h(a) \in \mathbb{C}$. Then we have the following partial generalization of Proposition 3.1 in [8].

Proposition 3.1. *The evaluation map $ev(n, d)_r$ is injective if and only if $r \leq d - 3$.*

Proof. Note that Proposition 3.1 in [8] implies the claim for $n = 2$. Suppose first that $r \leq d - 3$ and that the claim holds for $n - 1 \geq 2$. To fix the ideas, assume that $n = 2n_1 + 1$ is odd. Fix a_1 to be one of the local minimum points of T_d . The set of points in $\mathcal{N}(n, d)$ having the first coordinate equal to a_1 can be identified to the set of nodes $\mathcal{N}(n - 1, d)$, sitting in the affine space with coordinates x_2, \dots, x_n . If $h \in \ker ev(n, d)_r$, it follows that $h(a_1, -) \in \ker ev(n - 1, d)_r$. By our induction hypothesis, this kernel is trivial, hence $h(a_1, -) = 0$ in the polynomial ring $\mathbb{C}[x_2, \dots, x_n]$. It follows that h is divisible by the polynomial $x_1 - a_1$. Hence, in this way we get $N_1 \geq d/2$ linear factors of h of the form $x_1 - a_1$. Using all the coordinates, we'll get $N_n \geq nd/2 > d$ distinct linear factors for h . This implies that $h = 0$.

In the case $n = 2n_1$ even, we should take a_1 a local maximum point of T_d and all the rest goes in the same way as above.

To complete the proof it is enough to produce a polynomial $h \in \ker ev(n, d)$ with $\deg h = d - 2$. For this, let $f(n, d)(x_0, x_1, \dots, x_n)$ be the polynomial obtained from $g(n, d)$ by homogenization, in other words $f(n, d) = 0$ is an equation for $\mathcal{C}(n, d)$ in \mathbb{P}^n . We take

$$h(x_1, \dots, x_n) = \frac{\partial f(n, d)}{\partial x_0}(1, x_1, \dots, x_n).$$

This polynomial vanishes on $\mathcal{N}(n, d)$ by definition, and has degree $d - 2$ because in the Chebyshev polynomial $T_d(x)$ the monomial x^{d-1} is missing. □

Consider now the homogeneous ideal $I \subset S$ corresponding to the polynomials vanishing on the node set $\mathcal{N}(n, d)$. The above result is equivalent to $I_k = 0$ for $k < d - 2$ and $I_{d-2} \neq 0$. It follows that the corresponding defect

$$\text{def } S_k(\mathcal{N}(n, d)) = |\mathcal{N}(n, d)| - \dim S_k + \dim I_k$$

satisfies $\text{def } S_k(\mathcal{N}(n, d)) = |\mathcal{N}(n, d)| - \dim S_k$ for $k < d - 2$ and $\text{def } S_k(\mathcal{N}(n, d)) > |\mathcal{N}(n, d)| - \dim S_k$ for $k = d - 2$. Using Theorem 1.5 in [9] we get the following improvement of Corollary 9 in [3].

Corollary 3.2. *Let $\mathcal{C}(n, d)$ be the Chebyshev hypersurface of degree d in \mathbb{P}^n . Then the corresponding stability threshold is given by*

$$st(\mathcal{C}(n, d)) = T - (d - 3) = n(d - 2) + 1.$$

Moreover, the number of nodes is given by

- (i) $\tau(\mathcal{C}(n, d)) = \binom{2n_1}{n_1} d_1^n$ if $n = 2n_1$ is even and $d = 2d_1 + 1$ is odd;
- (ii) $\tau(\mathcal{C}(n, d)) = \binom{2n_1}{n_1} d_1^{n_1} (d_1 - 1)^{n_1}$ if $n = 2n_1$ is even and $d = 2d_1$ is even;
- (iii) $\tau(\mathcal{C}(n, d)) = \binom{2n_1+1}{n_1} d_1^{n_1+1} (d_1 - 1)^{n_1}$ if $n = 2n_1 + 1$ is odd and $d = 2d_1$ is even;
- (iii) $\tau(\mathcal{C}(n, d)) = \binom{2n_1+1}{n_1} d_1^n$ if $n = 2n_1 + 1$ is odd and $d = 2d_1 + 1$ is odd.

Note that the Chebyshev hypersurfaces are therefore among the very few classes of singular hypersurfaces D for which the exact value of the stability threshold $st(D)$ is known.

Concerning the coincidence thresholds for Chebyshev hypersurfaces we have the following.

Conjecture 3.3. *If $n = 2n_1$ is even, then $ct(\mathcal{C}(n, d)) = (n_1 + 1)d - n - 2$, i.e. the bounds given in Theorem 2.1 and Corollary 2.2 are best possible in this case.*

Example 3.4. For $n = 1$, the above Conjecture holds, see Example 2.3. For $n = 4$, if we compute with *Singular* the Hilbert-Poincaré series of the Milnor algebra $M(f(n, d))$ for small values of d , say $3 \leq d \leq 20$, we see that the coincidence threshold is given in this case by the expected formula $ct(\mathcal{C}(4, d)) = 3d - 6$. The same holds for $n = 6$, when $ct(\mathcal{C}(6, d)) = 4d - 8$.

4. ALEXANDER POLYNOMIALS OF NODAL HYPERSURFACES

Let D be a degree d hypersurface in \mathbb{P}^n , with $d \geq 2$ and $n \geq 1$, given by a reduced equation $f(x) = 0$. Consider the corresponding global Milnor fiber F defined by $f(x) - 1 = 0$ in \mathbb{C}^{n+1} with monodromy action $h : F \rightarrow F$, $h(x) = \exp(2\pi i/d) \cdot x$. When D has only isolated singularities, it is known that $\tilde{H}^k(F, \mathbb{C}) = 0$ for $k < n - 1$. The characteristic polynomial

$$(4.1) \quad \Delta_D(t) = \det(t \cdot I - h^* | H^{n-1}(F, \mathbb{C}))$$

is called the Alexander polynomial of the hypersurface D , with the convention $\Delta_D(t) = 1$ if $b_{n-1}(F) = 0$. To get a nontrivial Alexander polynomial $\Delta_D(t) \neq 1$, the idea is to look at hypersurfaces having lots of singularities, but sometime this is not enough, see Remark 4.6 (ii). That is why the number of examples with $\Delta_D(t) \neq 1$ is rather limited.

For nodal hypersurfaces, one has the following precise description, see Theorem (6.4.5) on page 208 in [5] and our formula (1.3).

Theorem 4.1. *Let D be a nodal hypersurface in \mathbb{P}^n of degree d and let \mathcal{N} its set of nodes. Then:*

- (i) $\Delta_D(t) = 1$ if nd is odd;

(ii) $\Delta_D(t) = [t + (-1)^{n+1}]^{\text{def } S_m(\mathcal{N})}$ if nd is even, where $m = nd/2 - n - 1$. In this second case, $\Delta_D(t) \neq 1$ if and only if

$$ct(D) < \frac{nd}{2} + d - n - 1 = \frac{T}{2} + \frac{d}{2}.$$

Remark 4.2. (i) For n even, the lower bound given by Corollary 2.2 can be written as

$$\frac{nd}{2} + d - n - 2 \leq ct(D).$$

It follows that in the case n even, $\Delta_D(t) \neq 1$ if and only if

$$ct(D) = \frac{nd}{2} + d - n - 2.$$

This seems to be the case for all odd dimensional Chebyshev hypersurfaces, see Conjecture 4.4, Example 4.5 and Remark 4.6. This explains our special attention devoted to this class of nodal hypersurfaces.

(ii) One may express the topological content of Theorem 4.1 using only Betti numbers as follows. If $n = 2n_1$ is even, then one has

$$b_n(D) = 1 + \text{def } S_m(\mathcal{N})$$

where $m = nd/2 - n - 1$. It follows that $\dim H^n(K^*(f))_{n_1d} = b_n(D) - 1$, i.e. $\text{mrd}(D) \geq n_1d - n$ and the number of nontrivial syzygies of the minimal expected degree $m = n_1d - n$ is determined topologically, exactly as in the case $n = 2$ covered by Theorem 4.1 in [9]. However, for $n > 2$, we do not have explicit formulas for these syzygies.

For n odd and d even, let D^2 be the double cover of \mathbb{P}^n ramified along D . Then

$$b_{n+1}(D^2) = 1 + \text{def } S_m(\mathcal{N}).$$

Example 4.3. Consider the *Kummer surface* S in \mathbb{P}^3 given by the affine equation

$$x^4 + y^4 + z^4 - y^2z^2 - z^2x^2 - x^2y^2 - x^2 - y^2 - z^2 + 1 = 0,$$

see for instance [13], p. 93. This surface has the maximum number of nodes for a surface in \mathbb{P}^3 of degree 4, namely 16 nodes. A direct computation with Singular yields

$$HP(M(f))(t) = 1 + 4t + 10t^2 + 16t^3 + 19t^4 + 16(t^5 + t^6 + \dots)$$

and hence for the Betti number of the associated 3-fold D^2 we get

$$b_4(D^2) = 1 + \text{def } S_2(\mathcal{N}) = 7.$$

Using Theorem 4.1 we may get a (potentially infinite) family of Chebyshev hypersurfaces in \mathbb{P}^n for $n = 3$ and $n = 4$ with rather large Alexander polynomials. We offer the following.

Conjecture 4.4. (i) Let $\mathcal{C}(3, d)$ be the Chebyshev surface of even degree $d = 2d_1$ in \mathbb{P}^3 . Then

$$\text{def } S_{3d_1-4}(\mathcal{N}(3, d)) = 3(d_1 - 1).$$

(ii) Let $\mathcal{C}(4, d)$ be the Chebyshev 3-folds of degree d in \mathbb{P}^4 . Then

$$\text{def } S_{2d-5}(\mathcal{N}(4, d)) = \left\lfloor \frac{d-1}{2} \right\rfloor \left(3 \left\lfloor \frac{d-1}{2} \right\rfloor - 1 \right).$$

Example 4.5. Conjecture 4.4 holds for all degrees d satisfying $3 \leq d \leq 20$.

As explained in the Example 3.4, for $n = 4$ the bounds given in Corollary 2.2 are best possible. In particular we have $\text{def } S_k(\mathcal{N}(4, d)) = 0$ for $k > 2d - 5$, but $\text{def } S_{2d-5}(\mathcal{N}(4, d)) > 0$. The actual values are computed using the Singular software.

In the case $n = 3$, the bounds given in Corollary 2.2 are not optimal. But we compute directly the Hilbert-Poincaré series of the Milnor algebra $M(f(n, d))$ in this range and use the relations between the coefficients and the defects $\text{def } S_k(\mathcal{N}(3, d))$ given by Theorem 1.5 in [9].

Remark 4.6. (i) It is interesting to note that in both cases above, one has

$$ct(D) = \frac{nd}{2} + d - n - 2,$$

i.e. the inequality in Theorem 4.1 is very tight in the case of Chebyshev hypersurfaces.

(ii) The feeling that the construction of nodal hypersurfaces D with $\Delta_D(t) \neq 1$ is not easy is reflected by the fact that in the case $n = 3$ all the even degree surfaces obtained from the Chebyshev surface by changing $k = 1$ into $k = -1$ in the equation (3.1) have $\Delta_D(t) = 1$, in spite of being nodal surfaces with lots of nodes.

(iii) Similar results hold for the Alexander polynomials of Chebyshev hypersurfaces of dimension ≥ 4 . We leave the interested reader to find the exact statements in each case. To do this, he has to compute the Hilbert-Poincaré series $HP(M(f))$ using a computer algebra software, check whether

$$a = \frac{nd}{2} + d - n - 1 - ct(D) > 0,$$

and then, in view of Theorem 1.5 in [9], compute the difference

$$\text{def } S_m(\mathcal{N}) = \dim M(f)_{ct(D)+a} - \dim M(f_s)_{ct(D)+a}.$$

Here $m = nd/2 - n - 1$ and $D_s : f_s = 0$ is any smooth hypersurface in \mathbb{P}^n of degree d .

REFERENCES

- [1] Arnold, V.I., Gusein-Zade, S.M., Varchenko, A.N., *Singularities of Differentiable Maps*. vols 1/2, Monographs in Math., **82/83**, Birkhäuser, Basel (1985/1988) 1
- [2] S.V.Chmutov, Examples of projective surfaces with many singularities. J.Algebraic Geom. 1 (1992), 191–196. 1
- [3] A. D. R. Choudary, A. Dimca, Koszul complexes and hypersurface singularities, Proc. Amer. Math. Soc. 121(1994), 1009–1016. 1, 2, 3
- [4] A. Dimca, On the Milnor fibrations of weighted homogeneous polynomials, Compositio Math. 76(1990), 19–47. 1
- [5] A. Dimca, *Singularities and Topology of Hypersurfaces*, Universitext, Springer-Verlag, 1992. 1, 1, 2, 4

- [6] A. Dimca, M. Saito, A generalization of Griffiths' theorem on rational integrals, *Duke Math. J.* 135(2006), 303–326. 1
- [7] A. Dimca, M. Saito, L. Wotzlaw, A generalization of Griffiths' theorem on rational integrals II, *Michigan Math. J.* 58(2009), 603–625. 1, 2, 2, 2, 2
- [8] A. Dimca, G. Sticlaru, Chebyshev curves, free resolutions and rational curve arrangements, *arXiv:1108.0798*. 3, 3
- [9] A. Dimca, G. Sticlaru, Koszul complexes and pole order filtrations, *arXiv:1108.3976*. 1, 1.1, 1, 2, 2, 2.3, 3, 4.2, 4, 4.6
- [10] D. Eisenbud, M. Green, and J. Harris, CayleyBacharach theorems and conjectures, *Bull. Amer. Math. Soc.* 33 (1996), 295–324. 1
- [11] D. Eisenbud, B. Ulrich, Regularity of the conductor, in preparation. 1
- [12] Ph. Griffiths, On the period of certain rational integrals I, II, *Ann. Math.* 90(1969), 460–541. 1
- [13] R.W.H.T. Hudson, *Kummer's quartic surface*, Cambridge University press, 1905. 4.3
- [14] R. Kloosterman, Cuspidal plane curves, syzygies and a bound on the MW-rank, *arXiv:1107.2043v2*. 1
- [15] A. Libgober, Alexander invariants of plane algebraic curves. *Proc. Symp. Pure Math.*, **40**, Part 2, 135–144 (1983). 1
- [16] A. Libgober, Homotopy groups of the complements to singular hypersurfaces, II, *Annals of Math.*, 139(1994), 117–144. 1
- [17] M. Saito, On b -function, spectrum and rational singularity, *Math. Ann.* 295(1993), 51–74. 2
- [18] CoCoA: a system for doing Computations in Commutative Algebra. Available at <http://cocoa.dima.unige.it> 1
- [19] Decker, W.; Greuel, G.-M.; Pfister, G.; Schönemann, H.: SINGULAR 3-1-3 — A computer algebra system for polynomial computations. <http://www.singular.uni-kl.de> (2011). 1

INSTITUT UNIVERSITAIRE DE FRANCE ET LABORATOIRE J.A. DIEUDONNÉ, UMR DU CNRS 6621, UNIVERSITÉ DE NICE SOPHIA-ANTIPOLIS, PARC VALROSE, 06108 NICE CEDEX 02, FRANCE
E-mail address: `dimca@unice.fr`

FACULTY OF MATHEMATICS AND INFORMATICS, OVIDIUS UNIVERSITY, BD. MAMAIA 124, 900527 CONSTANTA, ROMANIA
E-mail address: `gabrielsticlaru@yahoo.com`